FINITE-DIMENSIONAL CALOGERO REPRESENTATION OF THE q-DIFFERENTIAL OPERATOR

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A finite-dimensional matrix representation of the Jackson q-differential operator D_q, defined by $D_q f(x) = (f(qx) - f(x))/(x(q-1))$, is written down following Calogero. Such a representation of D_q should have applications in q-analysis leading to the corresponding extensions of the numerous results of Calogero's work.

1. Introduction

Recently Calogero[1] has extended to multidimensions his work on a convenient finite-dimensional matrix representation of the differential operator[2,3]. It may be recalled that the earlier findings in the one dimensional case led to several remarkable results related to matrix theory, integrable dynamical systems, classical polynomials, special functions, numerical treatment of Sturm-Liouville eigenvalue problems etc. (see Ref. 1 for detailed bibliography).

The purpose of the present short note is to write down the Calogero matrix representation of the Jackson q-differential operator in the one dimensional case, defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q-1)} = \left(\frac{q^{x\frac{d}{dx}} - 1}{x(q-1)} \right) f(x) \,. \tag{1.1}$$

We do this in view of the central role played by D_q in q-analysis (see, e.g., Refs. 4-6) and the recent interest in q-analysis in relation to the theory of quantum algebras, possible deformations of the current framework of quantum mechanics, etc., (see, e.g., Refs. 7-11). We hope that such a representation of D_q would lead to extensions of Calogero's remarkable results. We shall assume q to be generic throughout this note.

2. Calogero matrices for x and d/dx

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Calogero[1] gives a straightforward prescription for obtaining the required convenient finite-dimensional representation of any linear differential operator $\mathcal{A}(x, d/dx)$ within a chosen scheme of interpolation. Here, we shall adhere to Calogero's original scheme of the Lagrange interpolation[2,3]. In this section, we recall some of the basic formulae of Calogero which will help us write down, in the next section, the corresponding Calogero matrix representation for D_q.

With $\{f_j \mid j = 1, 2, \ldots, n\}$ denoting n given numbers, the 'interpolational function'

$$f(x) = \sum_{j=1}^{n} f_j \delta^{(j)}(x), \tag{2.1}$$

where $\left\{\delta^{(j)}(x) \mid j = 1, 2, \ldots, n\right\}$ are the 'interpolational polynomials'

$$\delta^{(j)}(x) = \prod_{k(\neq j)=1}^{n} \frac{x - x_k}{x_j - x_k}, \tag{2.2}$$

is such that $f(x)$ takes the values $\{f_j \mid j = 1, 2, \ldots, n\}$, respectively, at the chosen n distinct points ('nodes') $\{x_j \mid j = 1, 2, \ldots, n\}$:

$$f(x_j) = f_j, \qquad j = 1, 2, \ldots, n. \tag{2.3}$$

The Lagrange interpolational polynomials $\left\{\delta^{(j)}(x) \mid j = 1, 2, \ldots, n\right\}$ are linear combinations of the monomials $\{x^{j-1} \mid j = 1, 2, \ldots, n\}$ ('seeds' of the Lagrange interpolation), with the structure

$$\delta^{(j)}(x) = \sum_{k=1}^{n} C_{jk} x^{k-1}, \qquad j = 1, 2, \ldots, n, \tag{2.4}$$

where the matrix $C = [C_{jk}]$, with j as the row index and k as the column index, is the inverse of the Vandermonde matrix $V = [V_{jk}] = \left[x_j^{k-1}\right]$. Further, defining

$$\langle f, g \rangle = \sum_{j=1}^{n} f(x_j) g(x_j), \tag{2.5}$$

one has the orthogonality relation

$$\left\langle \delta^{(j)}, \delta^{(k)} \right\rangle = \delta_{jk}, \qquad j, k = 1, 2, \ldots, n. \tag{2.6}$$

Now, one has

$$\left\langle \delta^{(j)}, x \delta^{(k)} \right\rangle = x_j \delta_{jk}, \qquad j, k = 1, 2, \ldots, n., \tag{2.7}$$

or, in other words, the matrix

$$X = [X_{jk}] = [x_j \delta_{jk}] = \text{diag}(x_j), \qquad j, k = 1, 2, \ldots, n., \tag{2.8}$$

represents the operator 'multiplication by x' in the basis $\left\{\delta^{(j)} \mid j = 1, 2, \ldots, n\right\}$. This means that, as far as we are concerned with the values of functions at the distinct chosen nodes $\{x_j \mid j = 1, 2, \ldots, n\}$, the interpolational polynomials $\left\{\delta^{(j)} \mid j = 1, 2, \ldots, n\right\}$ are like

the Dirac delta functions and are 'eigenfunctions' of x in a sense. The corresponding Calogero matrix for the differential operator d/dx is obtained from the definition

$$D_{jk} = \left\langle \delta^{(j)}, \frac{d}{dx}\delta^{(k)} \right\rangle = \frac{d}{dx}\delta^{(k)} \bigg|_{x=x_j} . \tag{2.9}$$

It should be noted that the set $\left\{ \delta^{(j)} \mid j = 1, 2, \ldots, n \right\}$ is closed under differentiation, namely, the derivative of any $\delta^{(j)}$ can be expressed as a linear combination of $\left\{ \delta^{(k)} \mid k = 1, 2, \ldots, n \right\}$ with constant coefficients. This is a consequence of (2.4) and similar closure property of the set of seeds $\{x^{j-1} \mid j = 1, 2, \ldots n\}$ under differentiation. Explicitly,

$$D = BZB^{-1}, \tag{2.10}$$

where

$$B = \mathrm{diag}\,(b_j), \qquad b_j = \prod_{k(\neq j)=1}^{n} (x_j - x_k), \tag{2.11}$$

and

$$Z_{jk} = \begin{cases} (x_j - x_k)^{-1} & \text{if } j \neq k, \\[2mm] \sum_{k(\neq j)=1}^{n} (x_j - x_k)^{-1} & \text{if } j = k. \end{cases} \tag{2.12}$$

It is clear that for any other differential operator, $\mathcal{A}(x, x/dx)$, the corresponding Calogero matrix will be given by the correspondence rule

$$\mathcal{A}\left(x, \frac{d}{dx}\right) \rightarrow A = \mathcal{A}(X, D). \tag{2.13}$$

It follows from the earlier work of Calogero[2,3] that another useful form of the matrix D is:

$$D = X^{-1}VNV^{-1}, \tag{2.14}$$

where V is the Vandermonde matrix and

$$N = \mathrm{diag}(j - 1), \qquad j = 1, 2, \ldots, n, \tag{2.15}$$

provided $x_j \neq 0$ for any $j = 1, 2, \ldots, n$.

3. Calogero matrix for D_q

The first observation one can make with reference to the q-differential operator (1.1) is that the seeds $\{x^{j-1} \mid j = 1, 2, \ldots, n\}$ and hence the polynomials $\left\{ \delta^{(j)} \mid j = 1, 2, \ldots, n \right\}$ are closed under q-differentiation also:

$$D_q x^{j-1} = [j-1]_q x^{j-2}, \qquad j = 1, 2, \ldots, \tag{3.1}$$

with the Heine 'basic number' $[m]_q$ defined by

$$[m]_q = \frac{q^m - 1}{q - 1}. \tag{3.2}$$

Hence, it is clear that the Calogero matrix for D_q, say $\mathcal{D}$, is given by

$$\mathcal{D}_{jk} = \left\langle \delta^{(j)}, D_q \delta^{(k)} \right\rangle = \left. D_q \delta^{(k)}(x) \right|_{x=x_j} . \tag{3.3}$$

We shall assume that $x_j \neq 0$, for any $j = 1, 2, \ldots, n$. Then, using the prescription (2.13), along with (2.8) and (2.14), in (1.1), one can see easily that the explicit form of $\mathcal{D}$ is as follows:

$$\mathcal{D} = X^{-1} V [N]_q V^{-1}, \qquad [N]_q = \text{diag}\left([j-1]_q\right), \qquad j = 1, 2, \ldots, n, \tag{3.4}$$

where V is the Vandermonde matrix. In the limit $q \to 1$, $\mathcal{D}$ becomes Calogero's D in (2.14). As is obvious, the matrix $\mathcal{N} = X\mathcal{D}$ has the first n basic numbers, corresponding respectively to the first n nonnegative integers, as its eigenvalues independent of the choice of $\{x_j \mid j = 1, 2, \ldots, n\}$: this is the q-analogue of Calogero's result that the matrix XZ (or XD, since $Z = B^{-1}DB$ and B commutes with X) has the first n nonnegative integers as its eigenvalues independent of the choice of $\{x_j \mid j = 1, 2, \ldots, n\}$.

It may be noted that in q-analysis also, the matrix representing the operator 'multiplication by x' will be $X = \text{diag}(x_j)$, $j = 1, 2, \ldots, n$. Only wherever the differential operator d/dx has to be replaced by D_q, the Calogero matrix D in (2.14) will have to be replaced by its q-analogue $\mathcal{D}$ in (3.4).

4. Conclusion

To summarize, in this short note we have obtained the finite dimensional Calogero matrix representation of the q-differential operator (D_q) that would replace the finite dimensional Calogero matrix representation of the differential operator (d/dx) in studies on the q-analogues of the numerous results of Calogero relating to matrix theory, integrable dynamical systems, classical polynomials, special functions, and so on. The operator 'multiplication by x' would have the same Calogero matrix representation, namely, $x \to X = \text{diag}(x_j)$, with $j = 1, 2, \ldots, n$, in the q-analysis also (the only restriction we are required to have is that none of the nodes $\{x_j \mid j = 1, 2, \ldots, n\}$ should be at the origin).

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Note: After completion of this work, we have learnt that a closely related discussion of the 'finite dimensional representation of the Q operator' is to be found in a paper by Calogero and Ji Xiaoda with the title "Solvable (nonrelativistic, classical) n-body problems in multidimensions - II" due to appear in the Proceedings of a Meeting on Nonlinear Dynamics (Pavullo nel Frignano, Italy, May 1994) Eds. M. Costato, A. Degasperis and M. Milano (Publ.: Editrice Compositon, the Publishers of Nuovo Cimento).